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Density Nonlinearities and a Field Theory for the Dynamics of Simple Fluids

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We study the role of the Jacobian arising from a constraint enforcing the nonlinear relation $\mathbf{g} = \rho \mathbf{V}$, where ρ, $\mathbf{g}$, and $\mathbf{V}$ are the mass density, the momentum density, and the local velocity field, respectively, in the field-theoretic formulation of the nonlinear fluctuating hydrodynamics of simple fluids. By investigating the Jacobian directly and by developing a field-theoretic formulation without the constraint, we find that no changes in dynamics result as compared to the previous formulation developed by Das and Mazenko (DM). In particular, the cutoff mechanism discovered by DM is shown to be a consequence of the $1/\rho$ nonlinearity in the problem, not of the constraint. The consequences of this result for the static properties of the system are also discussed.

KEY WORDS: Field theory; nonlinear fluctuating hydrodynamics; idealized glass transition.

1. INTRODUCTION

The development of the appropriate field-theoretic treatment for nonlinear fluctuating hydrodynamics (NFH) of simple fluids is a more subtle enterprise than one might first imagine. If one develops a theory for simple fluids including the complete set of conserved fields, the mass density $\rho(\mathbf{x})$, the momentum density $\mathbf{g}(\mathbf{x})$, and the energy density $\varepsilon(\mathbf{x})$, then one finds[1] that one must include multiplicative noise[2] in order to gain consistency with thermodynamics. This surprising result is associated with the connection between the fluctuating energy, entropy, and temperature. In the case where the energy is not included in the set of slow variables, one still finds technical problems in developing the associated field theory. These

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problems are associated with the form of the kinetic energy density $g^2/2\rho$ resulting from Galilean invariance. This $1/\rho$ factor can be identified with the nonlinear relationship between the momentum density and the local velocity field: $\mathbf{V}(\mathbf{x}) = \mathbf{g}(\mathbf{x})/\rho(\mathbf{x})$ and, at first sight, seems to complicate the problem considerably.

In ref. 3 (DM), the NFH of compressible fluids was studied as a model for the glass transition. A field-theoretic formulation of the problem was developed by generalizing the standard Martin–Siggia–Rose (MSR) method[4] to include the nonlinear constraint between $\mathbf{g}(\mathbf{x})$ and $\mathbf{V}(\mathbf{x})$. In the functional integral formulation described in ref. 3, this constraint was enforced by introducing an auxiliary velocity field $\mathbf{V}(\mathbf{x})$ and inserting a delta-functional constraint,

$$\int \mathscr{D}\mathbf{V}(\mathbf{x})\,\delta(\mathbf{g} - \rho\mathbf{V}) \tag{1}$$

Enforcing the relation $\mathbf{g} = \rho\mathbf{V}$ eliminates the $1/\rho$ nonlinearity in the kinetic energy and yields a *polynomial* action in $\rho(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$, and $\mathbf{V}(\mathbf{x})$, which in turn allowed one to carry out a perturbation theory expansion with standard renormalization schemes. The effects due to nonlinearities on various physical quantities were then calculated at the one-loop order and, as a result of a systematic perturbative expansion, it was discovered that there is a nonhydrodynamic correction that cuts off the sharp nature of the idealized glass transition.[6],[2] This result has the important consequence that dense fluid systems remain ergodic for all values of the density and temperature, although the density feedback mechanism does drive the viscosity to large values.

Recently Schmitz *et al.*[8] suggested that a more faithful representation of the original Langevin equations requires that the constraint should be enforced by using, instead of Eq. (1), the identity

$$1 = \int \mathscr{D}\mathbf{V}(\mathbf{x})\,\delta\left(\frac{\mathbf{g}}{\rho} - \mathbf{V}\right) \tag{2}$$

$$= \|\rho\| \int \mathscr{D}\mathbf{V}(\mathbf{x})\,\delta(\mathbf{g} - \rho\mathbf{V}) \tag{3}$$

where $\|\rho\|$ is the Jacobian resulting from the change of variable in the delta functional. It was argued in ref. 8 that the cutoff mechanism found by DM may be an artifact resulting from their use of the contraint (1) rather than Eq. (2). Since these constraints differ only by the Jacobian factor $\|\rho\|$, this

[2] For a recent review of theoretical developments see ref. 7.

is equivalent to saying that the cutoff mechanism is eliminated if one includes the Jacobian factor into the development. We show here that this is not the case.

We can evaluate the Jacobian $\|\rho\|$ directly by expressing it in terms of an integral over Grassmann fields.[9] Inserting this result into the DM action, we will be able to show that it does not play any significant role on the dynamics. We then develop a similar functional integral formulation to the one in ref. 3, but without introducing any constraint. The $1/\rho$ nonlinearities are expanded as a power series in $\delta\rho = \rho - \rho_0$, and by comparing our perturbation theory results with those of ref. 3, we will be able to see explicitly the role of the Jacobian. We find that the two formulations are equivalent at the one-loop order of the perturbation theory and the Jacobian has no dynamical effects at this order. This seems perplexing, since it is clear that the Jacobian does influence the static equilibrium behavior generated by a general effective Hamiltonian. The resolution to this apprent contradiction is that the dynamics of a simple fluid, generated by a Langevin equation, is insensitive to changes in the chemical potential. It appears, at least to lowest order, that the effects of the Jacobian in question can be absorbed in the chemical potential and do not directly affect the statics generated by the Langevin equation.

In Section 2 we give a brief review of the formulation in ref. 3 just to collect some results needed for our discussion. In Section 3 the Jacobian is expressed as a Grassmann integral and the usefulness of this formulation is discussed. In Section 4 the theory is formulated without introducing the velocity field and the one-loop equivalence between the two formulations is studied in detail. The static theory is described in Section 5.

2. BRIEF REVIEW OF DM

Our starting point is the set of generalized Langevin equations for compressible fluids for the set of flow variables $\{\rho(\mathbf{x}), \mathbf{g}(\mathbf{x})\}$. Following standard procedure,[10] we obtain the continuity equation for the conservation of mass

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{g} \tag{4}$$

and the generalized Navier–Stokes equation for the conservation of momentum,

$$\frac{\partial g_i}{\partial t} = -\rho \nabla_i \frac{\delta F_u}{\delta \rho} - \sum_j \nabla_j \left(\frac{g_i g_j}{\rho}\right) - \sum_j L_{ij}\left(\frac{g_j}{\rho}\right) + \Theta_i \tag{5}$$

In Eq. (5), $F_u[\delta\rho]$ is the potential energy part of the effective Hamiltonian given by

$$F = F_K + F_u \tag{6}$$

where the kinetic energy F_K is

$$F_K = \int d^d\mathbf{x}\, \frac{\mathbf{g}^2}{2\rho} \tag{7}$$

In ref. 3, the simple choice of F_u,

$$F_u[\delta\rho] = \int d^d\mathbf{x}\, \frac{A}{2} (\delta\rho)^2 \tag{8}$$

was used, where $A^{-1} \equiv \rho_0/c_0^2$ is the flat static structure factor, ρ_0 is the average mass density, and c_0 is the bare sound speed. This simple quadratic part of F_u then gives the pressure nonlinearity in the Langevin equation which is responsible for the density feedback mechanism of the glass transition.[6] In general, F_u can be any local functional of $\delta\rho$ and the spatial derivatives of $\delta\rho$. By including the derivatives of $\delta\rho$, we can probe the effect of the spatial correlations in the system on the dynamics. The dissipative matrix in Eq. (5) is given by

$$L_{ij}(\mathbf{x}) = -\eta_0(\tfrac{1}{3}\nabla_i\nabla_j + \delta_{ij}\nabla^2) - \zeta_0\nabla_i\nabla_j \tag{9}$$

where η_0 is the bare shear viscosity and ζ_0 the bare bulk viscosity. For later use we define the bare longitudinal viscosity $\Gamma_0 = \zeta_0 + \frac{4}{3}\eta_0$. The noise Θ_i is Gaussian with variance

$$\langle \Theta_i(\mathbf{x}, t)\, \Theta_j(\mathbf{x}', t')\rangle = 2k_B T L_{ij}(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{x}')\, \delta(t - t') \tag{10}$$

These Langevin equations can be put into a field-theoretic form following the standard MSR procedure.[4,5,12] It essentially amounts to introducing a hatted variable $\hat{\psi}$ for each field $\psi = \rho$, $\mathbf{g}$ to enforce the equation of motion and integrating over the Gaussian noise to yield a quadratic action in $\hat{\psi}$. The generating functional without source terms is given by

$$Z = \int \mathscr{D}\psi\, \mathscr{D}\hat{\psi}\, e^{-S[\psi, \hat{\psi}]} \tag{11}$$

where the action $S[\psi, \hat{\psi}]$, with the notation $1 = (\mathbf{x}_1, t_1)$, is given by

$$S[\psi, \hat{\psi}] = \int d1 \left\{ \sum_{ij} \hat{g}_i \beta^{-1} L_{ij}(1)\, \hat{g}_j + i\hat{\rho}\left[\frac{\partial\rho}{\partial t_1} + \nabla_1 \cdot \mathbf{g}\right] \right.$$
$$\left. + i\sum_i \hat{g}_i \left[\frac{\partial g_i}{\partial t_1} + \rho\nabla_1^i \frac{\delta F_u}{\delta\rho} + \sum_j \nabla_1^j \left(\frac{g_i g_j}{\rho}\right) + \sum_j L_{ij}(1)\left(\frac{g_j}{\rho}\right)\right]\right\} \tag{12}$$

In ref. 3, the $1/\rho$ nonlinearities are removed by using Eq. (1) and the representation of a delta functional,

$$\delta(\mathbf{g}-\rho\mathbf{V})=\int \mathscr{D}\hat{\mathbf{V}} \exp\left\{i\int d1\, \hat{\mathbf{V}}(1)\cdot[\mathbf{g}(1)-\rho(1)\,\mathbf{V}(1)]\right\} \tag{13}$$

so that the generating functional in ref. 3 can be expressed as the functional integral over Ψ and $\hat{\Psi}$, where $\Psi=\rho, g_i, V_i$. Denoting hereafter the results from ref. 3 by the superscript I, we have

$$Z^{(\mathrm{I})}=\int \mathscr{D}\Psi\, \mathscr{D}\hat{\Psi}\, e^{-S^{(\mathrm{I})}[\Psi,\hat{\Psi}]} \tag{14}$$

where $S^{(\mathrm{I})}[\Psi, \hat{\Psi}]$ is given by

$$\begin{aligned} S^{(\mathrm{I})}[\Psi, \hat{\Psi}]=\int d1 &\left\{\sum_{ij} \hat{g}_i\beta^{-1}L_{ij}(1)\,\hat{g}_j+i\hat{\rho}\left[\frac{\partial\rho}{\partial t_1}+\nabla_1\cdot\mathbf{g}\right]\right.\\ &+i\sum_i \hat{g}_i\left[\frac{\partial g_i}{\partial t_1}+\rho\nabla_1^i\frac{\delta F_u}{\delta\rho}+\sum_j\nabla_1^j(\rho V_iV_j)+\sum_j L_{ij}(1)\,V_j\right]\\ &\left.+i\hat{\mathbf{V}}\cdot(\mathbf{g}-\rho\mathbf{V})\right\} \end{aligned} \tag{15}$$

The perturbative expansion for a polynomial action of this form is standard. The nonlinear corrections are given in the form of the self-energies $\Sigma_{\alpha\beta}$ modifying the inverse linear propagator $[G^0]^{-1}_{\alpha\beta}$ through Dyson's equation,

$$G^{-1}_{\alpha\beta}=[G^0]^{-1}_{\alpha\beta}-\Sigma_{\alpha\beta} \tag{16}$$

where $G_{\alpha\beta}=\langle\Psi_\alpha\Psi_\beta\rangle$. Detailed calculations of the self-energies were carried out in ref. 3. Here we list the results for the density response and the correlation functions. In the Fourier-transformed space,

$$G^{(\mathrm{I})}_{\rho\hat{\rho}}(\mathbf{q},\omega)=\frac{\rho(\mathbf{q},\omega)\omega+iL(\mathbf{q},\omega)}{D^{(\mathrm{I})}(\mathbf{q},\omega)} \tag{17}$$

$$G^{(\mathrm{I})}_{\rho\rho}(\mathbf{q},\omega)=\frac{2\beta^{-1}q^2\rho^2(\mathbf{q},\omega)\,\tilde{L}(\mathbf{q},\omega)}{|D^{(\mathrm{I})}(\mathbf{q},\omega)|^2} \tag{18}$$

where

$$\rho(\mathbf{q},\omega)=\rho_0-i\Sigma^{(\mathrm{I})}_{\hat{V}V}(\mathbf{q},\omega) \tag{19}$$

$$L(\mathbf{q},\omega)=q^2\Gamma_0+i\Sigma^{(\mathrm{I})}_{\hat{g}V}(\mathbf{q},\omega) \tag{20}$$

$$\tilde{L}(\mathbf{q},\omega)=q^2\Gamma_0-(2\beta^{-1})^{-1}\left\{\Sigma^{(\mathrm{I})}_{\hat{g}\hat{g}}(\mathbf{q},\omega)+\left(\frac{\Gamma_0q^2}{\rho_0}\right)^2\Sigma^{(\mathrm{I})}_{\hat{V}\hat{V}}(\mathbf{q},\omega)\right\} \tag{21}$$

and

$$D^{(1)}(\mathbf{q}, \omega) = \rho(\mathbf{q}, \omega)\{\omega^2 - q^2 c_0^2 - \Sigma_{g\rho}^{(1)}(\mathbf{q}, \omega)\} + iL(\mathbf{q}, \omega)\{\omega + iq\Sigma_{V\rho}^{(1)}(\mathbf{q}, \omega)\} \tag{22}$$

Here all the self-energies are their longitudinal parts. The density feedback mechanism is realized by calculating the one-loop self-energies contributing to the dynamic viscosities, $L(\mathbf{q}, \omega)$ and $\tilde{L}(\mathbf{q}, \omega)$, which are[3] quadratic in the density correlation function.[6] Without the nonhydrodynamic correction due to the self-energy $\Sigma_{V\rho}^{(1)}(\mathbf{q}, \omega)$, the response function is the same as the one that gives the density feedback mechanism. Thus $\Sigma_{V\rho}^{(1)}(\mathbf{q}, \omega)$ provides the DM cutoff. As mentioned in Section 1, and as indicated by the subscripts, the cutoff is related to the $1/\rho$ nonlinearities and the constraint between $\mathbf{g}$ and $\mathbf{V}$.

3. EVALUATION OF THE JACOBIAN

The question arises: what changes result if one uses the constraint (2) rather than Eq. (1) to introduce the velocity field? It should be clear from Eq. (3) that the only changes in the DM action come from the Jacobian factor $\|\rho\|$. This quantity can be represented as

$$\|\rho\| = \int \mathcal{D}\eta\, \mathcal{D}\bar{\eta} \exp\left[\int d1 \sum_i \bar{\eta}_i(1)\, \rho(1)\, \eta_i(1)\right] \tag{23}$$

where $\bar{\eta}_i$ and η_i are Grassmann fields. We must, as indicated below, be careful in using any unregularized representation of the Jacobian like Eq. (23) in the functional integral formalism, since it might not preserve causality. If we include Eq. (23) in the DM formulation, then the appropriate generating functional is given by

$$\tilde{Z} = \int \mathcal{D}\eta\, \mathcal{D}\bar{\eta}\, \mathcal{D}\Psi\, \mathcal{D}\hat{\Psi}\, e^{-\tilde{S}[\Psi, \hat{\Psi}, \eta, \bar{\eta}]} \tag{24}$$

where the action is given now by

$$\tilde{S}[\Psi, \hat{\Psi}, \eta, \bar{\eta}] = S^{(1)}[\Psi, \hat{\Psi}] + \int d1 \sum_i \bar{\eta}_i(1)\, \rho(1)\, \eta_i(1) \tag{25}$$

The only linear propagator involving $\bar{\eta}$ or η is

$$G^0_{\bar{\eta}_i \eta_j}(\mathbf{q}, \omega) = -G^0_{\eta_j \bar{\eta}_i}(\mathbf{q}, \omega) = \frac{\delta_{ij}}{\rho_0} \tag{26}$$

Fig. 1. The one-loop diagrams generated by the nonlinearity due to the Jacobian. The Jacobian is regularized such that the first diagram vanishes.

The new nonlinear term arising from the Grassmann fields is of the form, $\delta\rho \sum_i \bar{\eta}_i \eta_i$. This gives, at one-loop order, only two new self-energy diagrams (Fig. 1). The first appears to give a contribution to $\Sigma_{\rho\rho}$, which clearly violates causality. We note that the self-energies between two unhatted variables $\Sigma_{\psi\psi}$ are indeed equal to zero[3] in the original DM formulation in accordance with causality. Thus Eq. (23) has to be regularized. We note that if we use, instead of Eq. (23),

$$\|\rho\|_\varepsilon = \int \mathcal{D}\eta\, \mathcal{D}\bar{\eta} \exp\left[\int d1 \sum_i \bar{\eta}_i(1) \left\{\varepsilon \frac{\partial}{\partial t_1} + \rho(1)\right\} \eta_i(1)\right] \tag{27}$$

with a time derivative with a very small coefficient ε, then the first diagram of Fig. 1 vanishes, since it is in the form of a time integral of the product of a retarded and an advanced propagator. Furthermore, as will be clear later in this section, the regularization given as Eq. (27) guarantees causality to all orders of perturbation theory, i.e., $\Sigma_{\psi\psi}$ remains to vanish to all orders.

The second diagram in Fig. 1, however, does give a finite contribution. In the limit $\varepsilon \to 0$, we have

$$\Sigma_{\bar{\eta}_i\eta_j}(\mathbf{q}, \omega) = -\Sigma_{\eta_j\bar{\eta}_i}(\mathbf{q}, \omega) = -\delta_{ij} \frac{\beta^{-1}}{c_0^2} \delta_\Lambda(0) \equiv \delta_{ij}\Sigma \tag{28}$$

where

$$\delta_\Lambda(0) = \int^\Lambda \frac{d^d\mathbf{k}}{(2\pi)^d}$$

[3] The one-loop contributions to the self-energies $\Sigma_{\psi_\alpha\psi_\beta}$ between two unhatted variables come from either the cubic nonlinear terms $\hat{\psi}\psi\psi$ or the quartic nonlinear term $\hat{\psi}\psi\psi\psi$ in the DM action. The former generates a one-loop diagram that can expressed as a time integral of the product of an advanced and a retarded propagator, which vanishes. The contribution from the latter comes from the $\hat{\psi}\psi$ loop, which is exactly canceled by the Jacobian (which is unrelated to the one discussed in the present paper) originated intrinsically from the functional integral formulation of the MSR method. Therefore, at one-loop order, $\Sigma_{\psi_\alpha\psi_\beta} = 0$. This can be shown to be true to all orders.[12]

with the large-momentum cutoff Λ. This self-energy gives a correction to the correlation function between η and $\bar{\eta}$,

$$G_{\bar{\eta}_i\eta_j}(\mathbf{q},\omega)=\frac{\delta_{ij}}{\rho_0-\Sigma} \tag{29}$$

We note that there are no such self-energies at one-loop order that link the Grassmann fields to the ρ, $\mathbf{g}$, or $\mathbf{V}$. This fact, together with the vanishing of the first diagram in Fig. 1, shows that the inverse propagator matrix in Eq. (16) is in a block-diagram form with the $\Psi\hat{\Psi}$ block being identical to those given in DM formulation. Inverting the inverse propagator, we find that there are no changes, at one-loop order, in the response and correlation functions for ρ, $\mathbf{g}$, and $\mathbf{V}$ due to the Jacobian.

This result, in fact, can be generalized to all orders of perturbation theory by deriving the Ward identity from the invariance of the action, Eq. (25), under $\eta_i\to e^{i\sigma}\eta_i$ and $\bar{\eta}_i\to e^{-i\sigma}\bar{\eta}_i$ for any constant σ. Infinitesimally this symmetry transformation reads

$$\delta\eta_i=(i\sigma)\eta_i,\qquad \delta\bar{\eta}_i=(-i\sigma)\bar{\eta}_i,\qquad \delta\Psi_\alpha=\delta\hat{\Psi}_\alpha=0 \tag{30}$$

Introducing the sources J_α, $\hat{J}_\alpha$ for Ψ_α, $\hat{\Psi}_\alpha$ and the Grassmann sources $\bar{\xi}_i$, ξ_i for η_i, $\bar{\eta}_i$, respectively, into the generating functional, Eq. (24), we have

$$\begin{aligned}\tilde{Z}[J,\hat{J},\bar{\xi},\xi]=\int \mathscr{D}\eta\,\mathscr{D}\bar{\eta}\,\mathscr{D}\Psi\,\mathscr{D}\hat{\Psi}\exp\Bigg[&-\tilde{S}[\Psi,\hat{\Psi},\eta,\bar{\eta}]\\ &+\int d1\left(\sum_\alpha\{J_\alpha\Psi_\alpha+\hat{J}_\alpha\hat{\Psi}_\alpha\}+\sum_i\{\bar{\xi}_i\eta_i+\bar{\eta}_i\xi_i\}\right)\Bigg]\end{aligned} \tag{31}$$

Since the action $\tilde{S}$ and the integration measure in Eq. (31) are invariant under the transformation, Eq. (30), the variation of the source terms must vanish if we change the integration variable by Eq. (30). This yields the following Ward identity:

$$\int d1\sum_i\left\{\xi_i(1)\frac{\delta}{\delta\xi_i(1)}-\bar{\xi}_i(1)\frac{\delta}{\delta\bar{\xi}_i(1)}\right\}\tilde{Z}[J,\hat{J},\bar{\xi},\xi]=0 \tag{32}$$

Taking a derivative of Eq. (32) with respect to $\xi_j(3)$ and then with respect to $J_\alpha(2)$ and setting all the sources equal to zero, we have

$$0=\frac{\delta^2}{\delta J_\alpha(2)\,\delta\xi_j(3)}\tilde{Z}[J,\hat{J},\bar{\xi},\xi]\bigg|_{J=\hat{J}=\xi=\bar{\xi}=0}=-\tilde{Z}[0]\,G_{\Psi_\alpha\bar{\eta}_j}(2,3) \tag{33}$$

Similarly, we can easily derive

$$G_{\Psi\eta} = G_{\hat{\Psi}\eta} = G_{\hat{\Psi}\bar{\eta}} = 0 \tag{34}$$

It follows that, as in the one-loop case, the inverse propagator is block-diagonal. Let us now investigate whether the nonlinearity, $\delta\rho \sum_i \bar{\eta}_i \eta_i$ contributes to the self-energies in the $\Psi\hat{\Psi}$ block ($\Sigma_{\psi\psi}$, $\Sigma_{\psi\hat{\psi}}$, $\Sigma_{\hat{\psi}\psi}$, and $\Sigma_{\hat{\psi}\hat{\psi}}$). To do this, we first note that Eqs. (33) and (34) can be understood as a direct consequence of the charge conservation for η, $\bar{\eta}$ imposed by the symmetry, Eq. (30). We can imagine a charge flowing through the $\eta\bar{\eta}$ lines in the perturbative diagrams, say from η to $\bar{\eta}$. The vanishing of the correlation functions with only one external η or $\bar{\eta}$, Eqs. (33) and (34), then directly follows from the charge conservation. The internal $\eta\bar{\eta}$ lines must then form a complete circle following the charge flow. The most general diagram involving η, $\bar{\eta}$ that can contribute to the self-energies in the $\Psi\hat{\Psi}$ block is drawn in Fig. 2. At one-loop order, it is just the first diagram in Fig. 1, which vanishes because of the regularization, Eq. (27). We have, in fact, the same situation for the general diagram in Fig. 2. This diagram is expressed in terms of the integral over l loop momenta and frequencies, where l is the number of the loops in the diagram. Now we inverse Fourier transform the zeroth-order $\eta\bar{\eta}$ propagators to get the time integral over $t_1,..., t_n$, where n is the number of $G^0_{\eta\bar{\eta}}(t_i)$'s in the diagram. Then we can integrate over one frequency, which yields the delta function $\delta(\sum_i t_i)$; therefore we have the following factor inside the integral over l momenta and $l-1$ frequencies:

$$\int \prod_{i=1}^{n} dt_i \, \delta\left(\sum_{i=1}^{n} t_i\right) G^0_{\eta\bar{\eta}}(t_1) \cdots G^0_{\eta\bar{\eta}}(t_n) = 0 \tag{35}$$

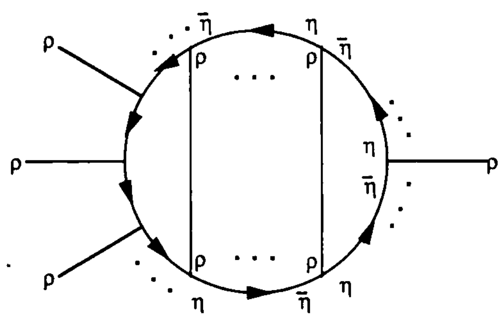

Fig. 2. The most general diagram generated by the nonlinearity due to the Jacobian without external Grassmann fields. The arrow indicates the charge flow.

which vanishes due to the regularization, Eq. (27), since all the $G^0_{\eta\bar{\eta}}(t_i)$'s are advanced propagators. This shows that the nonlinearity involving the Grassmann fields does not affect the self-energies in the $\Psi\hat{\Psi}$ block, especially $\Sigma_{\psi\psi}=0$, even after including the regularized Jacobian, Eq. (27). We can therefore conclude that the Jacobian, included in the DM action as an integral over the Grassmann fields, as in Eq. (25), has no effect on the correlation and response functions for ρ, $\mathbf{g}$, and $\mathbf{V}$ at all orders of perturbation theory.

4. FORMULATION WITHOUT V FIELDS

Schmitz *et al.*[8] have suggested that the cutoff mechanism found by DM may be somehow introduced artificially through the implementation of the constraint condition. By looking at the original theory without the constraint within perturbation theory, we show that this worry is without foundation. Starting from the original action (12), we expand $1/\rho$ as a power series in $\delta\rho$. Keeping nonlinear terms that are relevant to the one-loop order, we have in the action, Eq. (12),

$$\sum_j \nabla_j\left(\frac{g_i g_j}{\rho}\right) \cong \sum_j \nabla_j\left(\frac{g_i g_j}{\rho_0}\right) - \sum_j \nabla_j\left(\frac{g_i g_j(\delta\rho)}{\rho_0^2}\right) + \cdots \tag{36}$$

$$\sum_j L_{ij}\left(\frac{g_j}{\rho}\right) \cong \sum_j L_{ij}\left(\frac{g_j}{\rho_0}\right) - \sum_j L_{ij}\left(\frac{g_j(\delta\rho)}{\rho_0^2}\right) + \sum_j L_{ij}\left(\frac{g_j(\delta\rho)^2}{\rho_0^3}\right) + \cdots \tag{37}$$

Let us denote the results of this formulation by the superscript II. Only three kinds of self-energies are generated by the nonlinearities: $\Sigma^{(\mathrm{II})}_{\hat{g}_i\rho}(\mathbf{q},\omega)$, $\Sigma^{(\mathrm{II})}_{\hat{g}_i g_j}(\mathbf{q},\omega)$, and $\Sigma^{(\mathrm{II})}_{\hat{g}_i\hat{g}_j}(\mathbf{q},\omega)$. Using these self-energies, we can represent various correlation and response functions. For example,

$$G^{(\mathrm{II})}_{\rho\hat{\rho}}(\mathbf{q},\omega) = \frac{\omega + i(\Gamma_0/\rho_0)q^2 - \Sigma^{(\mathrm{II})}_{\hat{g}g}(\mathbf{q},\omega)}{D^{(\mathrm{II})}(\mathbf{q},\omega)} \tag{38}$$

$$G^{(\mathrm{II})}_{\rho\rho}(\mathbf{q},\omega) = \frac{q^2\{2\beta^{-1}\Gamma_0 q^2 - \Sigma^{(\mathrm{II})}_{\hat{g}\hat{g}}(\mathbf{q},\omega)\}}{|D^{(\mathrm{II})}(\mathbf{q},\omega)|^2} \tag{39}$$

where

$$D^{(\mathrm{II})}(\mathbf{q},\omega) = \omega^2 - q^2c_0^2 - \Sigma^{(\mathrm{II})}_{\hat{g}\rho}(\mathbf{q},\omega) + i\frac{\Gamma_0}{\rho_0}q^2\omega - \omega\Sigma^{(\mathrm{II})}_{\hat{g}g}(\mathbf{q},\omega) \tag{40}$$

We note that the number of self-energies is reduced from seven to three compared to the previous case, since only two variables, ρ and $\mathbf{g}$, are considered here. But the number of diagrams we have to consider for each self-energy is increased according to the appearance of the new nonlinearities in Eqs. (36) and (37). At one-loop order, by comparing the nonlinear vertex structures of the two actions, Eq. (12) with the expansions (36), (37) and Eq. (15), and by using the relations among the linear propagators

$$G^0_{\rho\hat{V}}(\mathbf{q}, \omega) = \frac{\Gamma_0 q^2}{\rho_0} G^0_{\rho\hat{g}}(\mathbf{q}, \omega) \tag{41}$$

$$G^{0,L}_{g\hat{V}}(\mathbf{q}, \omega) = \frac{\Gamma_0 q^2}{\rho_0} G^{0,L}_{g\hat{g}}(\mathbf{q}, \omega) \tag{42}$$

$$G^{0,L}_{V\hat{V}}(\mathbf{q}, \omega) = \frac{i}{\rho_0} + \frac{\Gamma_0 q^2}{\rho_0^2} G^{0,L}_{g\hat{g}}(\mathbf{q}, \omega) \tag{43}$$

with Γ_0 being replaced by η_0 for transverse propagators, we find that the new diagrams for each self-energy of (II) are in one-to-one correspondence to the diagrams of the corresponding self-energies of (I) that are not present in formation (II). For example, in Fig. 3, the first five diagrams

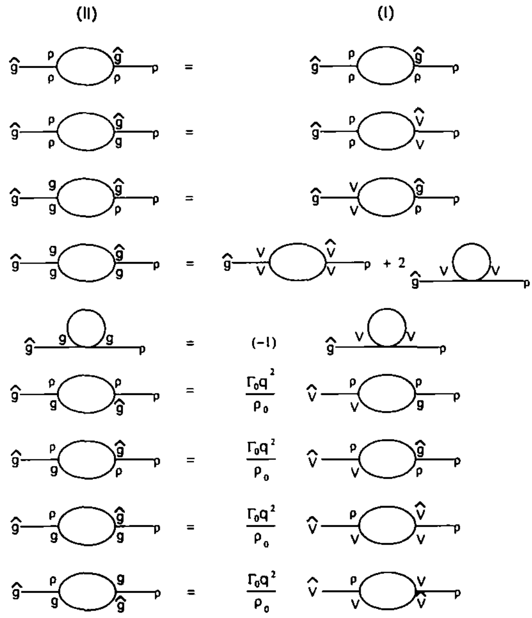

Fig. 3. The detailed correspondence between the one-loop diagrams contributing to $\Sigma^{(II)}_{\hat{g}\rho}$ and $\Sigma^{(I)}_{\hat{g}\rho} + (\Gamma_0 q^2/\rho_0)\, \Sigma^{(I)}_{\hat{V}\rho}$. See Eq. (44).

contributing to $\Sigma_{\hat{g}\rho}^{(II)}$ reproduce the corresponding diagrams of $\Sigma_{\hat{g}\rho}^{(I)}$. The remaining diagrams contain the $\hat{g}g\rho$ vertex from Eq. (37) with $\hat{g}$ on external legs, which give us a factor of $\Gamma_0\, q^2/\rho_0$ for each diagram. Other than this factor, we find that these are the diagrams of the self-energy $\Sigma_{\hat{V}\rho}^{(I)}$. Thus, at one-loop order, we have

$$\Sigma_{\hat{g}\rho}^{(II)}(\mathbf{q}, \omega) = \Sigma_{\hat{g}\rho}^{(I)}(\mathbf{q}, \omega) + \frac{\Gamma_0 q^2}{\rho_0}\, \Sigma_{\hat{V}\rho}^{(I)}(\mathbf{q}, \omega) \tag{44}$$

Similarly, as seen from Figs. 4 and 5, we find that

$$\Sigma_{\hat{g}g}^{(II)}(\mathbf{q}, \omega) = \frac{1}{\rho_0}\, \Sigma_{\hat{g}V}^{(I)}(\mathbf{q}, \omega) + \frac{\Gamma_0 q^2}{\rho_0^2}\, \Sigma_{\hat{V}V}^{(I)}(\mathbf{q}, \omega) \tag{45}$$

$$\begin{aligned} \Sigma_{\hat{g}\hat{g}}^{(II)}(\mathbf{q}, \omega) &= \Sigma_{\hat{g}\hat{g}}^{(I)}(\mathbf{q}, \omega) + \frac{\Gamma_0 q^2}{\rho_0}\, \Sigma_{\hat{g}\hat{V}}^{(I)}(\mathbf{q}, \omega) + \frac{\Gamma_0 q^2}{\rho_0}\, \Sigma_{\hat{V}\hat{g}}^{(I)}(\mathbf{q}, \omega) \\ &\quad + \left(\frac{\Gamma_0 q^2}{\rho_0}\right)^2 \Sigma_{\hat{V}\hat{V}}^{(I)}(\mathbf{q}, \omega) \\ &= \Sigma_{\hat{g}\hat{g}}^{(I)}(\mathbf{q}, \omega) + \left(\frac{\Gamma_0 q^2}{\rho_0}\right)^2 \Sigma_{\hat{V}\hat{V}}^{(I)}(\mathbf{q}, \omega) \end{aligned} \tag{46}$$

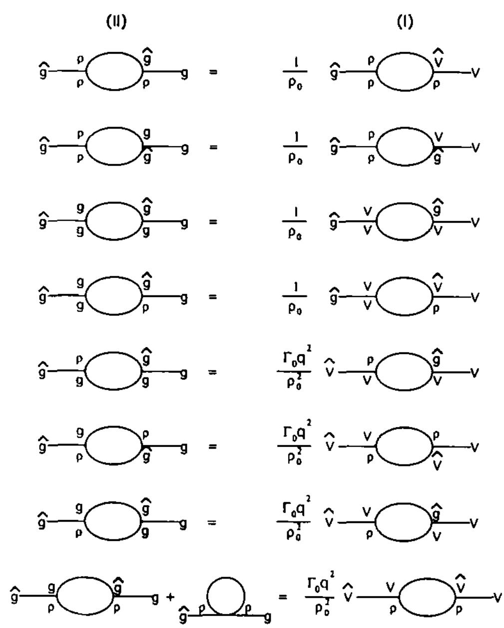

Fig. 4. The detailed correspondence between the one-loop diagrams contributing to the self-energy in Eq. (45).

since $\Sigma^{(1)}_{\hat{g}\hat{V}}(\mathbf{q},\omega) = -\Sigma^{(1)}_{\hat{V}\hat{g}}(\mathbf{q},\omega)$. Using these relations, we can express the response and correlation functions in formulation (II), Eqs. (38) and (39), in terms of the self-energies in formulation (I). We first note that, since we are dealing with one-loop corrections, we can neglect quadratic and higher-order terms in Σ's. Inserting Eqs. (44) and (45) into Eq. (40), we have

$$D^{(II)}(\mathbf{q},\omega) = \rho^{-1}(\mathbf{q},\omega)\, D^{(I)}(\mathbf{q},\omega) \tag{47}$$

From Eqs. (38), (39) and using Eqs. (44)–(46) and (47), we have, at one-loop order,

$$G^{(II)}_{\rho\hat{\rho}}(\mathbf{q},\omega) = G^{(I)}_{\rho\hat{\rho}}(\mathbf{q},\omega), \qquad G^{(II)}_{\rho\rho}(\mathbf{q},\omega) = G^{(I)}_{\rho\rho}(\mathbf{q},\omega) \tag{48}$$

which shows that the two formulations (I) and (II) are *equivalent* to first order. In particular, the DM cutoff is recovered, as seen from the second term in the right-hand side of Eqs. (44) and (47). Therefore the cutoff is generated not by a particular form of a constraint condition, but by the intrinsic $1/\rho$ nonlinearity in the problem.

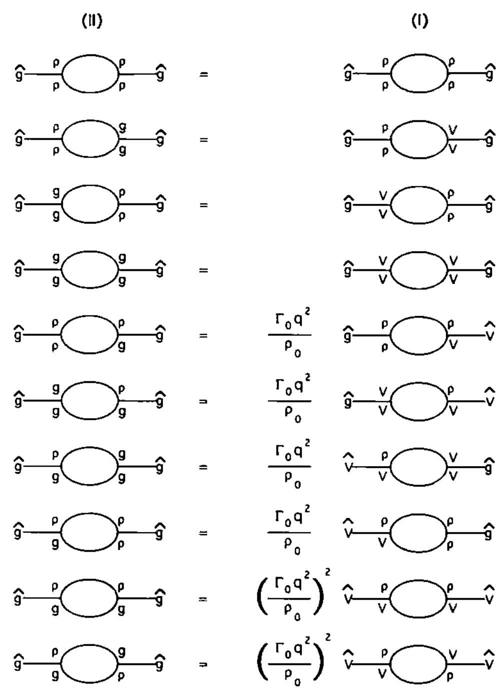

Fig. 5. The detailed correspondence between the one-loop diagrams contributing to the self-energies in Eq. (46).

The renormalizations of various hydrodynamic quantities for the two formulations are also equivalent. For example, the renormalized sound speed, for both (I) and (II), is given by

$$c^2 = c_0^2 + \lim_{\mathbf{q},\omega \to 0} \frac{1}{q} \Sigma_{\dot{g}\rho}(\mathbf{q}, \omega) \tag{49}$$

But, because of Eq. (44), $\lim(1/q)\Sigma^{(\mathrm{I})}_{\dot{g}\rho} = \lim(1/q)\Sigma^{(\mathrm{II})}_{\dot{g}\rho}$. This quantity was evaluated in ref. 3 to show that

$$\lim_{\mathbf{q},\omega \to 0} \frac{1}{q} \Sigma^{(\mathrm{I})}_{\dot{g}\rho}(\mathbf{q}, \omega) = 0 \tag{50}$$

i.e., there is no one-loop correction to the hydrodynamic sound speed.

5. STATIC THEORY

We have investigated in Sections 3 and 4 the role of the Jacobian in the dynamics governed by the Langevin equations, and found, to lowest order, that there are no changes in the dynamics due to the Jacobian. Let us now look at the static equilibrium behavior. The static equal-time averages of the fields ψ_i are calculated with respect to the effective Hamiltonian, Eq. (6):

$$\langle \psi_i \psi_j \rangle = \int \mathscr{D}\psi \, e^{-\beta F} \psi_i \psi_j / Z \tag{51}$$

where $Z = \int \mathscr{D}\psi \, e^{-\beta F}$. Nonlinear corrections come from the $1/\rho$ factor in the kinetic energy, Eq. (7). We note that the potential energy part of the effective Hamiltonian F_u appears in the Langevin equation only in the form of $\rho \nabla^i (\delta F_u / \delta \rho)$. Therefore a chemical potential-like term in F_u, which is linear in ρ, does not affect the Langevin equation. But, in the statics, such a term controls the renormalizations of the average $\langle \rho \rangle$ of the mass density and the sound speed c. To be specific, let us take the simple example

$$F_u = \int d^d\mathbf{x} \left[\frac{A}{2} \rho^2 - \mu \rho \right] \tag{52}$$

which gives the same Langevin equation as Eq. (8). The one-loop [i.e., $O(\beta^{-1})$] corrections to $\langle \rho \rangle$ can be calculated by expanding $\rho = \langle \rho \rangle + \Delta\rho$ and integrating out the **g** field. As a result, we have the following one-loop effective action in $\Delta\rho$:

$$F_{\mathrm{eff}}[\Delta\rho] = \frac{1}{2} \left\{ A + \beta^{-1} \frac{d\delta_\Lambda(0)}{2\langle \rho \rangle^2} \right\} (\Delta\rho)^2 + \left\{ A\langle \rho \rangle - \mu - \beta^{-1} \frac{d\delta_\Lambda(0)}{2\langle \rho \rangle} \right\} (\Delta\rho) \tag{53}$$

Then $\langle\rho\rangle$ is determined by setting the linear term in $\Delta\rho$ to zero. Thus we have

$$\langle\rho\rangle=\rho_0+\beta^{-1}\rho_1+O(\beta^{-2}) \tag{54}$$

$$\rho_1=\frac{\mu_1}{A}+\frac{d\delta_A(0)}{2A\rho_0} \tag{55}$$

where $\mu=A\rho_0+\beta^{-1}\mu_1+O(\beta^{-2})$. The sound speed is given by the usual thermodynamic result,

$$c^2=\beta^{-1}\langle\rho\rangle\langle\Delta\rho\,\Delta\rho\rangle^{-1} \tag{56}$$

The derivation of Eq. (56) from the Langevin equation follows using the Fokker–Planck description of the problem.[11] The one-loop correction to the static density–density correlation function can easily be read off from the quadratic term of the effective action, Eq. (53):

$$\beta\langle\Delta\rho\,\Delta\rho\rangle=\frac{1}{A}-\beta^{-1}\frac{d\delta_A(0)}{2A^2\rho_0^2}+O(\beta^{-2}) \tag{57}$$

Therefore,

$$c^2=c_0^2+\beta^{-1}\left(\mu_1+\frac{d\delta_A(0)}{\rho_0}\right)+O(\beta^{-2}) \tag{58}$$

This is consistent with the dynamics result, Eq. (50), if we choose $\mu_1=-d\delta_A(0)/\rho_0$. Thus, in some sense, the system *selects* the corresponding static limit as it evolves through the Langevin equations. The above discussion indicates that any effect the Jacobian might have on the static renormalization can be absorbed in the chemical poitentional μ chosen by the Langevin equation.

6. CONCLUSION

We have shown that the Jacobian, Eq. (3), arising from the constraint enforcing the nonlinear relation between **g** and **V** does not change the dynamics of Das and Mazenko. In particular, the functional integral formulations with and without a constraint are shown to be equivalent at least to lowest order. Therefore the cutoff mechanism discovered in ref. 3 is a genuine effect of the $1/\rho$ nonlinearity in the problem and not an artifact of the particular form of the constraint condition.

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